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Evaluation of Singular Electric Field Integral Equation (EFIE) Matrix Elements

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The electric field integral equation (EFIE) solution for scattering from an arbitrary three-dimensional geometry was published by Rao, Wilton, and Glisson (RWG) in 1982. For electrically large problems, with large systems of equations, the physical fidelity of the solution can be improved with increased numerical accuracy of the Galerkin inner products which populate the method of moments (MoM) impedance matrix. A new treatment of these inner products is presented here which improves the accuracy of the singular surface integrals that result from the Galerkin formulation.

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Contents

Introduction 1

Galerkin Solution of the EFIE 1

Regularization of Singular Surface Integrals 3

Simplex Coordinates 4

Decomposition of the Domain 4

Common Facet 8

Duffy Transformation 11

Analytic Integration over ξ_1, ξ_2 13

Analytic Integration over ω 15

Common Edge 16

Analytic Integration over ξ_2 19

Common Vertex 20

Conclusion 23

References 24

Evaluation of Singular Electric Field Integral Equation (EFIE) matrix elements

Introduction

The electric field integral equation (EFIE) solution for scattering from an arbitrary three-dimensional geometry was outlined in a paper by Rao, Wilton and Glisson (RWG) in 1982 [1]. Since that time their formulation has been incorporated into many computer codes to solve a large variety of scattering and radiation problems, e.g. EIGER, Carlos3D, FISC, Feko, MoM3D, IBC3D [2-7]. Experience has taught us that the accuracy of quantities derived from the original RWG solution are a function of the electrical size of the problem, which is reflected in the order of the system of equations that must be solved, and the electrical dimensions of the triangular patches [8].

The ill-conditioned matrices that result from electrically small patches can be remedied using of the 'loop-star' approach [9]. For electrically large problems, with large order systems of equations, the physical fidelity of the solution depends on the numerical accuracy of the Galerkin inner products which populate the method of moments (MoM) impedance matrix. A new treatment of these inner products is presented here which improves the accuracy of the singular surface integrals that result from the Galerkin formulation.

The question of accuracy has been a subject of research in the computational mechanics field for some time. Recently, the computational mechanics community has begun formulating boundary integral equation (BIE) solutions of Fredholm integral equations using a Galerkin descretization [10,11]. The focus of their research was to obtain high accuracy in the numerical treatment of the resulting singular 4-dimensional integrals. The methods developed for computational mechanics can be directly applied to the Galerkin (RWG) solution of the EFIE to obtain high accuracy in the numerical integration of the singular surface integrals that are fundamental to the numerical solution.

Galerkin Solution of Electromagnetic EFIE

The traditional RWG solution will be summarized here for objects that are represented by perfectly conducting surfaces. The time dependence is given by $e^{j\omega t}$. The electromagnetic problem is formulated using the electric field integral equation (EFIE) which is then used to compute unknown electric currents, $\vec{J}$, on the surface of the object. These surface currents are integrated to produce the desired quantity, for example, the far field scattering amplitude.

The unknown currents are expanded in terms of RWG vector basis functions on the surface. A single RWG basis set element, shown in Figure 1, consists of a pair of triangular patches that share a common edge. The Galerkin solution of the EFIE generates a system of linear equations which has the amplitudes of the RWG current

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expansion as unknowns. This system of equations can be written in the following compact form,

$$\mathbf{Z}\vec{J} = \vec{V}. \tag{1}$$

The square matrix $\mathbf{Z}$ is dense and complex valued. This system of equations is solved using standard numerical linear algebra methods. The vector $\vec{V}$ represents the electric field excitation on the structure, usually a plane wave. The individual elements of the $\mathbf{Z}$ matrix are inner products of an RWG basis and testing function with the EFIE operator for the PEC surface boundary condition. These inner products generate double integrals over pairs of RWG basis functions and appear as a sum of 4 terms,

$$Z_{jk} = \sum_{p,q=\pm 1} Z_{jk}^{p,q}. \tag{2}$$

Each term in the sum represents the interaction of one triangle with another in the RWG basis and testing function pair. This interaction is computed using the following expression,

$$Z_{jk}^{p,q} = jk\eta \frac{S_j^p S_k^q}{A_j^p A_k^q} l_j l_k \int dA_j^p \int dA_k^q \left(\frac{\vec{\rho}_j^{\,p} \cdot \vec{\rho}_k^{\,q}}{4} - \frac{1}{k^2} \right) g(\vec{r}, \vec{r}'), \tag{3}$$

where $g(\vec{r},\vec{r}')$ is the free space Greens function,

$$g(\vec{r},\vec{r}') = \frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}, \tag{4}$$

and,

$$j = \sqrt{-1},$$

$$k = \frac{2\pi f}{c_0}, c_0 = \text{speed of light}, \; f = \text{wave frequency},$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \text{impedance of free space},$$

$$A_j^p, A_k^q = \text{area of each triangle},$$

$$\vec{\rho}_j^{\,p}, \vec{\rho}_k^{\,q} = \text{vector from vertex opposite edge } j, k,$$

$$S_j^p, S_k^q = \pm 1, \text{current flow toward or away from edge},$$

$$l_j, l_k = \text{edge lengths associated with RWG triangle basis}.$$

A numerical difficulty arises in evaluating (3) over regions where the denominator of the Greens function becomes zero. When this occurs, the weak singularity in the integrand of the integral must be interpreted in a Cauchy principle value sense.

The treatment of this singularity has generally consisted of employing a singularity extraction approach where the Greens function in the integrand is split into two parts,

$$g(\vec{r},\vec{r}') = \frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{e^{-jk|\vec{r}-\vec{r}'|}-1}{4\pi|\vec{r}-\vec{r}'|} + \frac{1}{4\pi|\vec{r}-\vec{r}'|}. \tag{5}$$

This produces two integrals, the first has a regular integrand, which is numerically integrated, and the second that is evaluated analytically [12]. This approach is limited because the integrand still contains a R^{-1} term to be integrated numerically, requiring a large number of evaluations for a given accuracy. This method has also been applied to formulations with curved geometry where the singularity extraction is the first term in a Taylor's series, however this approximate method can be improved upon with the formulation outlined here.

Erichsen and Andra [10,11] encountered integrals of the type shown in (3), and others with higher order singularities, and were able to completely remove the singularity using a combination of relative coordinates, changing the order of integration and Duffy's transformations [13]. In addition to completely removing the singularity their method also allowed some parts of the 4-dimensional Galerkin integrals to be evaluated analytically, thereby reducing the effort required to numerically evaluate the entire inner product. Their technique for regularizing multi-dimensional singular surface integrals is applied here to the traditional RWG EFIE solution.

Regularization of Singular Surface Integrals

The integrals in (3) can be categorized into 5 classes. The criterion is based upon the geometrical relationship between the two triangular regions, labeled A_j^p, A_k^q. The fundamental distinction is whether or not the two triangles intersect in some manner, and if they do not, then how close or far apart are the two triangles. The 5 classes are summarized in the following table.

Type	*Relationship*	*Distribution of Singularity*
1	Common facet	two dimensional planar
2	Common edge	one dimensional line
3	Common vertex	single point
4	Distance < Facet Dimension	Near Singular
5	Distance > Facet Dimension	Non-Singular

Types 1 through 3 have singularities in the integrands and explicit formulas will be developed to analytically remove the singularity. Type 4 integrals are described as near singular and are studied elsewhere [14]. Type 5 present no singular behavior and can be evaluated using standard numerical methods for integrating regular functions over a triangular region [15].

The procedure for transforming (3) into a regular, non-singular, surface integral involves the following steps.

1. Introduce relative coordinates.
2. Represent the domain of integration such that the outer integration is over the relative coordinates.
3. Recombine symmetric integration domains.
4. Use Duffy's coordinates to remove the weak singularity in the remaining singular integrals.
5. Employ ordinary gaussian quadrature on the remaining regular integrals.

Simplex Coordinates

The integral in (3) is evaluated by transforming from ordinary 3-dimensional Cartesian coordinates to a simplex coordinate system. In simplex coordinates each triangle is mapped to a standard unit triangle. The unit triangle is defined by,

$$(\xi_1, \xi_2): 0 < \xi_1 < 1,\ \ 0 < \xi_2 < \xi_1 . \tag{6}$$

A triangle in 3-dimensional Cartesian space with vertices, $\vec{V}_1, \vec{V}_2, \vec{V}_3$, is mapped onto this unit triangle such that,

$$\begin{aligned} \vec{V}_1 &\Rightarrow (\xi_1, \xi_2) = (0,0) \\ \vec{V}_2 &\Rightarrow (\xi_1, \xi_2) = (1,0) \\ \vec{V}_3 &\Rightarrow (\xi_1, \xi_2) = (1,1) \end{aligned} \tag{7}$$

Any point in the triangle defined with three vertices can now be described using the 2 simplex coordinates.

Ignoring the constants outside the integral shown in (3) the transformed integral using two sets of simplex coordinates is,

$$\int\int\left(\frac{\vec{\rho}_j^{\,p}\cdot\vec{\rho}_k^{\,q}}{4}-\frac{1}{k^2}\right)g(\vec{r},\vec{r}\,')dA_j^p dA_k^q = 2A_j^p 2A_k^q \int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1}\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}d\eta_2 d\eta_1 d\xi_2 d\xi_1, \tag{8}$$

where the integrand has now been written as ratio of a regular function divided by a distance function. The constants in front of the new integral are the Jacobian of the transform from Cartesian to simplex coordinates.

Decomposition of the domain

The 4-dimensional Galerkin inner product integral is decomposed into six separate 4-dimensional integrals in the process of converting to relative coordinates and interchanging the order of integration. Explicit details of the process are presented here because the method allows for variations, for example curved triangular patches or rectangular patches, to be analyzed using the same approach.

The fundamental quantity to evaluate is the Galerkin inner product integral. Using simplex coordinates, $\overline{\eta},\overline{\xi}$, the integral is converted to the following standard form,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1}\frac{f(\overline{\xi},\overline{\eta})}{R(\overline{\xi},\overline{\eta})}d\eta_2 d\eta_1 d\xi_2 d\xi_1. \tag{9}$$

The order of integration for ξ_2 and η_1 is interchanged in preparation of a conversion to the relative coordinates,

$$\int_{\xi_1=0}^{1}\int_{\eta_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1}\frac{f(\overline{\xi},\overline{\eta})}{R(\overline{\xi},\overline{\eta})}d\eta_2 d\xi_2 d\eta_1 d\xi_1. \tag{10}$$

Relative coordinates are introduced next, the relative coordinates are formed from the simplex coordinates in the following manner,

$$\begin{aligned} u_1 &= \eta_1 - \xi_1, \\ u_2 &= \eta_2 - \xi_2. \end{aligned} \tag{11}$$

The integral using relative coordinates as the integration variables is now written in the following form,

$$\int_{\xi_1=0}^{1}\int_{u_1=-\xi_1}^{1-\xi_1}\int_{\xi_2=0}^{\xi_1}\int_{u_2=-\xi_2}^{u_1+\xi_1-\xi_2}\frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})}du_2 d\xi_2 du_1 d\xi_1. \tag{12}$$

The domain of integration of this integral is shown in Figure 2. The goal is to interchange the order of integration in the integral so that the outer integrations are over the relative coordinates u_1 and u_2.

Examining the integration domain in Figure 2 facilitates the interchange of order of integration. By inspection the ξ_1, u_1 integral upon interchange becomes,

$$\int_{\xi_1=0}^{1}\int_{u_1=-\xi_1}^{1-\xi_1}\cdots du_1 d\xi_1 = \int_{u_1=-1}^{0}\int_{\xi_1=-u_1}^{1}\cdots d\xi_1 du_1 + \int_{u_1=0}^{1}\int_{\xi_1=0}^{1-u_1}\cdots d\xi_1 du_1. \tag{13}$$

Note that the original domain, a single trapezoid, has now been divided into two triangular regions. The ξ_2, u_2 integration interchange produces three new integrals,

$$\int_{\xi_2=0}^{\xi_1}\int_{u_2=-\xi_2}^{u_1+\xi_1-\xi_2}\cdots du_2 d\xi_2 = \int_{u_2=-\xi_1}^{0}\int_{\xi_2=-u_2}^{\xi_1}\cdots d\xi_2 du_2 + \int_{u_2=0}^{u_1}\int_{\xi_2=0}^{\xi_1}\cdots d\xi_2 du_2 + \int_{u_2=u_1}^{u_1+\xi_1}\int_{\xi_2=0}^{\xi_1+u_1-u_2}\cdots d\xi_2 du_2. \tag{14}$$

The original domain of integration has been divided into two triangular and one rectangular region.

Collecting together the six new integrals that result from the initial interchange of integration variables produces the following set of terms,

$$\mathrm{I}^1 = \int_{u_1=-1}^{0}\int_{\xi_1=-u_1}^{1}\int_{u_2=-\xi_1}^{0}\int_{\xi_2=-u_2}^{\xi_1} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 du_2 d\xi_1 du_1, \tag{15}$$

$$\mathrm{I}^2 = \int_{u_1=-1}^{0}\int_{\xi_1=-u_1}^{1}\int_{u_2=0}^{u_1}\int_{\xi_2=0}^{\xi_1} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 du_2 d\xi_1 du_1, \tag{16}$$

$$\mathrm{I}^3 = \int_{u_1=-1}^{0}\int_{\xi_1=-u_1}^{1}\int_{u_2=u_1}^{u_1+\xi_1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 du_2 d\xi_1 du_1, \tag{17}$$

$$\mathrm{I}^4 = \int_{u_1=0}^{1}\int_{\xi_1=0}^{1-u_1}\int_{u_2=-\xi_1}^{0}\int_{\xi_2=-u_2}^{\xi_1} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 du_2 d\xi_1 du_1, \tag{18}$$

$$\mathrm{I}^5 = \int_{u_1=0}^{1}\int_{\xi_1=0}^{1-u_1}\int_{u_2=0}^{u_1}\int_{\xi_2=0}^{\xi_1} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 du_2 d\xi_1 du_1, \tag{19}$$

$$\mathrm{I}^6 = \int_{u_1=0}^{1}\int_{\xi_1=0}^{1-u_1}\int_{u_2=u_1}^{u_1+\xi_1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 du_2 d\xi_1 du_1. \tag{20}$$

The next step is to interchange the order of the, ξ_1, u_2, integration in the six integrals. Starting with I^1, the interchange results in two terms,

$$\int_{\xi_1=-u_1}^{1}\int_{u_2=-\xi_1}^{0} \cdots du_2 d\xi_1 = \int_{u_2=-1}^{u_1}\int_{\xi_1=-u_2}^{1} \cdots d\xi_1 du_2 + \int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1} \cdots d\xi_1 du_2 . \tag{21}$$

The completed interchange of integration variables now produces a split I^1 term,

$$\begin{aligned} \mathrm{I}^1 = \mathrm{I}^1_1 + \mathrm{I}^1_2 = &\int_{u_1=-1}^{0}\int_{u_2=-1}^{u_1}\int_{\xi_1=-u_2}^{1}\int_{\xi_2=-u_2}^{\xi_1} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 d\xi_1 du_2 du_1 \\ &+ \int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=-u_2}^{\xi_1} \frac{f(\overline{\xi},\overline{\mu})}{R(\overline{\xi},\overline{\mu})} d\xi_2 d\xi_1 du_2 du_1 . \end{aligned} \tag{22}$$

The result for I^2 is,

$$\mathrm{I}^2 = \int_{u_1=-1}^{0}\int_{u_2=0}^{u_1}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=0}^{\xi_1}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1, \tag{23}$$

I^3 is similar to I^1 because it produces two terms,

$$\begin{aligned}\mathrm{I}^3 = \mathrm{I}_1^3 + \mathrm{I}_2^3 &= \int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{f(\bar{\xi},\bar{\mu})}{R(\bar{\xi},\bar{\mu})}d\xi_2 d\xi_1 du_2 du_1\\ &+\int_{u_1=-1}^{0}\int_{u_2=0}^{1+u_1}\int_{\xi_1=u_2-u_1}^{1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{f(\bar{\xi},\bar{\mu})}{R(\bar{\xi},\bar{\mu})}d\xi_2 d\xi_1 du_2 du_1.\end{aligned} \tag{24}$$

Interchanging the order of integration for the remaining, I^4, I^5 and I^6 integrals produce only a single term in each case,

$$\mathrm{I}^4 = \int_{u_1=0}^{1}\int_{u_2=u_1-1}^{0}\int_{\xi_1=-u_2}^{1-u_1}\int_{\xi_2=-u_2}^{\xi_1}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1, \tag{25}$$

$$\mathrm{I}^5 = \int_{u_1=0}^{1}\int_{u_2=0}^{u_1}\int_{\xi_1=0}^{1-u_1}\int_{\xi_2=0}^{\xi_1}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1, \tag{26}$$

$$\mathrm{I}^6 = \int_{u_1=0}^{1}\int_{u_2=u_1}^{0}\int_{\xi_1=u_2-u_1}^{1-u_1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1. \tag{27}$$

The final step is to combine integrals together with overlapping domains. The naming convention here is used to reproduce the work of Andra,

$$\begin{aligned}\mathrm{E}_1 = \mathrm{I}_2^1 + \mathrm{I}^2 + \mathrm{I}_1^3 &= \int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\left\{\int_{\xi_2=-u_2}^{\xi_1} + \int_{\xi_2=\xi_1}^{0} + \int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\right\}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1\\ &= \int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=-u_2}^{\xi_1-(u_2-u_1)}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1\end{aligned} \tag{28}$$

$$\mathrm{E}_2 = \mathrm{I}^5 = \int_{u_1=0}^{1}\int_{u_2=0}^{u_1}\int_{\xi_1=0}^{1-u_1}\int_{\xi_2=0}^{\xi_1}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1 \tag{29}$$

$$\mathrm{E}_3 = \mathrm{I}_2^3 = \int_{u_1=-1}^{0}\int_{u_2=0}^{1+u_1}\int_{\xi_1=u_2-u_1}^{1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{f(\bar{\xi},\bar{\mu})}{R(\bar{\xi},\bar{\mu})}d\xi_2 d\xi_1 du_2 du_1 \tag{30}$$

$$\mathrm{E}_4 = \mathrm{I}^4 = \int_{u_1=0}^{1}\int_{u_2=u_1-1}^{0}\int_{\xi_1=-u_2}^{1-u_1}\int_{\xi_2=-u_2}^{\xi_1}\frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})}d\xi_2 d\xi_1 du_2 du_1 \tag{31}$$

$$E_5 = I_1^1 = \int_{u_1=-1}^{0}\int_{u_2=-1}^{u_1}\int_{\xi_1=-u_2}^{1}\int_{\xi_2=-u_2}^{\xi_1} \frac{f(\bar{\xi},\bar{\mu})}{R(\bar{\xi},\bar{\mu})} d\xi_2 d\xi_1 du_2 du_1 \tag{32}$$

$$E_6 = I^6 = \int_{u_1=0}^{1}\int_{u_2=u_1}^{1}\int_{\xi_1=u_2-u_1}^{1-u_1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)} \frac{f(\bar{\xi},\bar{u})}{R(\bar{\xi},\bar{u})} d\xi_2 d\xi_1 du_2 du_1 \tag{33}$$

These six integrals, E_1 through E_6, represent the domain decomposition of the original single domain integral.

Common Facet

The Galerkin inner product for this type involves integrating over identical triangular facets. The evaluation of the inner product integral depends on the functional form of the singularity in simplex coordinates. The Greens function spatial distance term,

$$R = |\vec{r} - \vec{r}'|, \tag{34}$$

expressed in the transform coordinates becomes,

$$\begin{aligned}
\vec{r} &= (1-\xi_1)\vec{V}_1 + (\xi_1 - \xi_2)\vec{V}_2 + \xi_1\vec{V}_3 \\
\vec{r}' &= (1-\eta_1)\vec{V}_1 + (\eta_1 - \eta_2)\vec{V}_2 + \eta_1\vec{V}_3 \\
\vec{r} - \vec{r}' &= (\eta_1 - \xi_1)\vec{V}_1 + (\eta_2 - \xi_2 - \eta_1 + \xi_1)\vec{V}_2 + (\xi_1 - \eta_1)\vec{V}_3 \\
&= u_1\vec{V}_1 + (u_2 - u_1)\vec{V}_2 - u_1\vec{V}_3 \\
&= u_1(\vec{V}_1 - \vec{V}_2 - \vec{V}_3) + u_2\vec{V}_2
\end{aligned} \tag{35}$$

and is a function of the relative coordinates only. The scalar distance is,

$$\begin{aligned}
R &= \sqrt{\left(u_1(\vec{V}_1 - \vec{V}_2 - \vec{V}_3) + u_2\vec{V}_2\right)\cdot\left(u_1(\vec{V}_1 - \vec{V}_2 - \vec{V}_3) + u_2\vec{V}_2\right)}, \\
&= \sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2},
\end{aligned} \tag{36}$$

and can be expressed in terms of a simple polynomial in the relative coordinates only. The coefficients are,

$$\begin{aligned}
\alpha_1 &= (\vec{V}_1 - \vec{V}_2 - \vec{V}_3)\cdot(\vec{V}_1 - \vec{V}_2 - \vec{V}_3), \\
\alpha_2 &= (\vec{V}_1 - \vec{V}_2 - \vec{V}_3)\cdot\vec{V}_2, \\
\alpha_3 &= \vec{V}_2 \cdot \vec{V}_2.
\end{aligned} \tag{37}$$

This allows the Greens function portion to be moved outside the inner two integrals in the inner product.

The remaining parts of the integrand involve dot products of vectors in the plane of the triangle, each vector is defined with a starting point in a prescribed vertex of the triangle and an ending point that resides in the triangle. Even though the case considered here is for identical facets, or triangles, the two vectors in the dot product are not necessarily the same. In 3-dimensional Cartesian space let these two vectors be denoted by,

$$\begin{aligned}\vec{\rho}_j &= \vec{r} - \vec{v}_j, \\ \vec{\rho}_k &= \vec{r}\,' - \vec{v}_k.\end{aligned} \tag{38}$$

Expressing these vectors in terms of the simplex coordinates for the most general case,

$$\begin{aligned}\vec{\rho}_j &= \vec{r} - \vec{v}_j = (1-\xi_1)\vec{\mathrm{V}}_1 + (\xi_1 - \xi_2)\vec{\mathrm{V}}_2 + \xi_1\vec{\mathrm{V}}_3 - \vec{v}_j, \\ \vec{\rho}_k &= \vec{r}\,' - \vec{v}_k = (1-\eta_1)\vec{\mathrm{V}}_1' + (\eta_1 - \eta_2)\vec{\mathrm{V}}_2' + \eta_1\vec{\mathrm{V}}_3' - \vec{v}_k.\end{aligned} \tag{39}$$

The dot product generates the following expression,

$$\vec{\rho}_j \cdot \vec{\rho}_k = \left((1-\xi_1)\vec{\mathrm{V}}_1 + (\xi_1 - \xi_2)\vec{\mathrm{V}}_2 + \xi_1\vec{\mathrm{V}}_3 - \vec{v}_j\right)\cdot\left((1-\eta_1)\vec{\mathrm{V}}_1' + (\eta_1 - \eta_2)\vec{\mathrm{V}}_2' + \eta_1\vec{\mathrm{V}}_3' - \vec{v}_k\right) \tag{40}$$

Expanding this and simplifying the resulting expression,

$$\begin{aligned}\vec{\rho}_j \cdot \vec{\rho}_k &= \left((1-\xi_1)\vec{\mathrm{V}}_1 + (\xi_1 - \xi_2)\vec{\mathrm{V}}_2 + \xi_1\vec{\mathrm{V}}_3 - \vec{v}_j\right)\cdot\left((1-\eta_1)\vec{\mathrm{V}}_1' + (\eta_1 - \eta_2)\vec{\mathrm{V}}_2' + \eta_1\vec{\mathrm{V}}_3' - \vec{v}_k\right) \\ &= \left(\xi_1(\vec{\mathrm{V}}_2 + \vec{\mathrm{V}}_3 - \vec{\mathrm{V}}_1) - \xi_2\vec{\mathrm{V}}_2 + \vec{\mathrm{V}}_1 - \vec{v}_j\right)\cdot\left(\eta_1(\vec{\mathrm{V}}_2' + \vec{\mathrm{V}}_3' - \vec{\mathrm{V}}_1') - \eta_2\vec{\mathrm{V}}_2' + \vec{\mathrm{V}}_1' - \vec{v}_k\right) \\ &= \beta_1\xi_1\eta_1 + \beta_2\xi_1\eta_2 + \beta_3\xi_2\eta_1 + \beta_4\eta_2\xi_2 + \beta_5\xi_1 + \beta_6\xi_2 + \beta_7\eta_1 + \beta_8\eta_2 + \beta_9\end{aligned} \tag{41}$$

where again the function can be written as a polynomial. The coefficients of the polynomial are given by,

$$\begin{aligned}\beta_1 &= (\vec{\mathrm{V}}_2 + \vec{\mathrm{V}}_3 - \vec{\mathrm{V}}_1)\cdot(\vec{\mathrm{V}}_2' + \vec{\mathrm{V}}_3' - \vec{\mathrm{V}}_1') \\ \beta_2 &= -(\vec{\mathrm{V}}_2 + \vec{\mathrm{V}}_3 - \vec{\mathrm{V}}_1)\cdot\vec{\mathrm{V}}_2' \\ \beta_3 &= (\vec{\mathrm{V}}_2 + \vec{\mathrm{V}}_3 - \vec{\mathrm{V}}_1)\cdot(\vec{\mathrm{V}}_1' - \vec{v}_k) \\ \beta_4 &= \beta_2 \\ \beta_5 &= \vec{\mathrm{V}}_2\cdot\vec{\mathrm{V}}_2' \\ \beta_6 &= -\vec{\mathrm{V}}_2\cdot(\vec{\mathrm{V}}_1' - \vec{v}_k) \\ \beta_7 &= (\vec{\mathrm{V}}_1 - \vec{v}_j)\cdot(\vec{\mathrm{V}}_2' + \vec{\mathrm{V}}_3' - \vec{\mathrm{V}}_1') \\ \beta_8 &= -(\vec{\mathrm{V}}_1 - \vec{v}_j)\cdot\vec{\mathrm{V}}_2' \\ \beta_9 &= (\vec{\mathrm{V}}_1 - \vec{v}_j)\cdot(\vec{\mathrm{V}}_1' - \vec{v}_k)\end{aligned} \tag{42}$$

The same dot product in relative coordinates is,

$$\vec{\rho}_j \cdot \vec{\rho}_k = \beta_1\xi_1(u_1+\xi_1)+\beta_2\xi_1(u_2+\xi_2)+\beta_3\xi_2(u_1+\xi_1)+\beta_4(u_2+\xi_2)\xi_2 \\ +\beta_5\xi_1+\beta_6\xi_2+\beta_7(u_1+\xi_1)+\beta_8(u_2+\xi_2)+\beta_9 \tag{43}$$

Collecting together the expressions for the Greens function and the dot product the original integrand becomes,

$$\left(\frac{\vec{\rho}_j^{\,p}\cdot\vec{\rho}_k^{\,q}}{4}-\frac{1}{k^2}\right)g(\vec{r},\vec{r}\,')=\frac{1}{4}h(\overline{\xi},\overline{u})g(\overline{u}). \tag{44}$$

It is this function that will be used as the integrand for the six integrals that span the domain of integration of the original inner product. However, the six integrals can be paired together to form three integrals because of the symmetry of the integration domains. This pairing of integrals with symmetric domains will simplify the problem. First consider the integrals E_1 and E_2,

$$E_1=\int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=-u_2}^{\xi_1-(u_2-u_1)}\frac{1}{4}h(\overline{\xi},\overline{u})g(\overline{u})d\xi_2 d\xi_1 du_2 du_1, \tag{45}$$

$$E_2=\int_{u_1=0}^{1}\int_{u_2=0}^{u_1}\int_{\xi_1=0}^{1-u_1}\int_{\xi_2=0}^{\xi_1}\frac{1}{4}h(\overline{\xi},\overline{u})g(\overline{u})d\xi_2 d\xi_1 du_2 du_1. \tag{46}$$

Let $\overline{\xi}=\overline{\xi}'-\overline{u}$ and $\overline{u}=-\overline{z}$ in E_1. The results are,

$$\begin{aligned} E_1 &= \int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=-u_2}^{\xi_1-(u_2-u_1)}\frac{1}{4}h(\overline{\xi},\overline{u})g(\overline{u})d\xi_2 d\xi_1 du_2 du_1 \\ &= \int_{z_1=1}^{0}\int_{z_2=z_1}^{0}\int_{\xi_1=z_1}^{1}\int_{\xi_2=z_2}^{\xi_1+(z_2-z_1)}\frac{1}{4}h(\overline{\xi},-\overline{z})g(-\overline{z})d\xi_2 d\xi_1 dz_2 dz_1 \\ &= \int_{z_1=0}^{1}\int_{z_2=0}^{z_1}\int_{\xi_1'=0}^{1-z_1}\int_{\xi_2'=0}^{\xi_1}\frac{1}{4}h(\overline{\xi}'+\overline{z},-\overline{z})g(-\overline{z})d\xi_2' d\xi_1' dz_2 dz_1 \end{aligned} \tag{47}$$

The limits of integration are identical to that of E_2 so that the two integrals can be combined together,

$$J_1=E_1+E_2=\int_{u_1=0}^{1}\int_{u_2=0}^{u_1}\int_{\xi_1=0}^{1-u_1}\int_{\xi_2=0}^{\xi_1}\frac{1}{4}h(\overline{\xi},\overline{u})g(\overline{u})+\frac{1}{4}h(\overline{\xi}+\overline{u},-\overline{u})g(-\overline{u})d\xi_2 d\xi_1 du_2 du_1. \tag{48}$$

The Greens function is an even function of the argument,

$$g(-\overline{u})=g(\overline{u}), \tag{49}$$

the new integral, J_1, becomes,

$$\mathrm{J}_1 = \mathrm{E}_1 + \mathrm{E}_2 = \int_{u_1=0}^{1}\int_{u_2=0}^{u_1}\int_{\xi_1=0}^{1-u_1}\int_{\xi_2=0}^{\xi_1}\frac{1}{4}\left\{h(\bar{\xi},\bar{u}) + h(\bar{\xi}+\bar{u},-\bar{u})\right\}g(\bar{u})d\xi_2 d\xi_1 du_2 du_1 . \tag{50}$$

The integrals E_3 and E_4 can be combined together in a similar fashion. Substitute $\bar{\xi} = \bar{\xi}' - \bar{u}$ and $\bar{u} = -\bar{z}$ in E_3,

$$\begin{aligned}\mathrm{E}_3 &= \int_{u_1=-1}^{0}\int_{u_2=0}^{1+u_1}\int_{\xi_1=u_2-u_1}^{1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{1}{4}h(\bar{\xi},\bar{u})g(\bar{u})d\xi_2 d\xi_1 du_2 du_1 \\ &= \int_{z_1=0}^{1}\int_{z_2=z_1-1}^{0}\int_{\xi_1'=-z_2}^{1-z_1}\int_{\xi_2'=0}^{\xi_1'}\frac{1}{4}h(\bar{\xi}'+\bar{z},-\bar{z})g(-\bar{z})d\xi_2' d\xi_1' dz_2 dz_1\end{aligned} \tag{51}$$

This is combined with E_4,

$$\mathrm{E}_4 = \int_{u_1=0}^{1}\int_{u_2=u_1-1}^{0}\int_{\xi_1=-u_2}^{1-u_1}\int_{\xi_2=-u_2}^{\xi_1}\frac{1}{4}h(\bar{\xi},\bar{u})g(\bar{u})d\xi_2 d\xi_1 du_2 du_1 \tag{52}$$

to produce,

$$\mathrm{J}_2 = \mathrm{E}_3 + \mathrm{E}_4 = \int_{u_1=0}^{1}\int_{u_2=u_1-1}^{0}\int_{\xi_1=-u_2}^{1-u_1}\int_{\xi_2=-u_2}^{\xi_1}\frac{1}{4}\left\{h(\bar{\xi},\bar{u}) + h(\bar{\xi}+\bar{u},-\bar{u})\right\}g(\bar{u})d\xi_2 d\xi_1 du_2 du_1 . \tag{53}$$

Lastly, combine E_5 and E_6 together by substituting $\bar{\xi} = \bar{\xi}' - \bar{u}$ and $\bar{u} = -\bar{z}$ in E_5,

$$\mathrm{J}_3 = \mathrm{E}_5 + \mathrm{E}_6 = \int_{u_1=0}^{1}\int_{u_2=u_1}^{1}\int_{\xi_1=u_2-u_1}^{1-u_1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{1}{4}\left\{h(\bar{\xi},\bar{u}) + h(\bar{\xi}+\bar{u},-\bar{u})\right\}g(\bar{u})d\xi_2 d\xi_1 du_2 du_1 . \tag{54}$$

At this point the original integral has been successfully transformed into a sum of three integrals with the desired order of integration, this result can be written simply as,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1}\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}d\eta_2 d\eta_1 d\xi_2 d\xi_1 = \sum_{i=1}^{3}\mathrm{J}_i . \tag{55}$$

However, the singularity in the Greens function is still present. The next step is to employ a series of Duffy coordinate transformations that will eliminate the singularity. The removal of the singularity will also convert the three integrals into one regular integral.

Duffy Transformations

The Duffy coordinate transform is used to remove a singularity at the origin in two dimensional and higher order integrals. Each of the J integrals has a singularity at the origin, namely $\bar{u} = 0$. The (u_1,u_2) coordinate integration domains as shown in Figure 3 for each of the J_i integrals.

In each of the J integrals the Greens function contains a singularity that is a function of the relative coordinates only. In relative coordinates, the Greens function for the common facet configuration is,

$$g(\bar{u}) = \frac{e^{jk\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}}}{4\pi\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}}. \tag{56}$$

The J_1 integral over the relative coordinates uses the standard two-dimensional Duffy transform. Consider the part of J_1 that is the integral of the Greens function over the relative coordinates only,

$$\int_{u_1=0}^{1}\int_{u_2=0}^{u_1} \frac{e^{jk\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}}}{4\pi\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}} \cdots du_2 du_1 . \tag{57}$$

Make the substitution,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega\eta \end{pmatrix}, \tag{58}$$

of these coordinates into the integral,

$$\int_{u_1=0}^{1}\int_{u_2=0}^{u_1} \frac{e^{jk\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}}}{4\pi\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}} \cdots du_2 du_1 = \int_{\omega=0}^{1}\int_{\eta=0}^{1} \frac{e^{jk\omega\sqrt{\alpha_1 + 2\alpha_2\eta + \alpha_3\eta^2}}}{4\pi\omega\sqrt{\alpha_1 + 2\alpha_2\eta + \alpha_3\eta^2}}\, \omega \cdots d\eta d\omega . \tag{59}$$

The singularity has been converted into a single term, ω, in the denominator that is exactly canceled out by the Jacobian of the coordinate transformation in the numerator. The resulting non-singular integral is,

$$J_1 = \int_{\omega=0}^{1}\int_{\eta=0}^{1} \frac{e^{jk\omega\sqrt{\alpha_1 + 2\alpha_2\eta + \alpha_3\eta^2}}}{4\pi\sqrt{\alpha_1 + 2\alpha_2\eta + \alpha_3\eta^2}} \cdots d\eta d\omega. \tag{60}$$

However, J_2 is not in the proper form for a direct Duffy coordinate transform,

$$J_2 = \int_{u_1=0}^{1}\int_{u_2=u_1-1}^{0} \cdots du_2 du_1, \tag{61}$$

but can be put converted to standard form by substituting $z = u_1 - u_2$,

$$J_2 = \int_{u_1=0}^{1} \int_{z=u_1}^{1} \cdots \, dz du_1 . \tag{62}$$

The Duffy transform for this type is,

$$\begin{aligned} u_1 &= \eta\omega, \\ z &= \omega, \end{aligned} \tag{63}$$

converting back to the original relative coordinates produces the necessary transform,

$$\begin{aligned} u_1 &= \eta\omega, \\ z = u_1 - u_2 &= \omega, \\ u_2 = u_1 - \omega &= \omega(\eta - 1). \end{aligned} \tag{64}$$

Using this transform in the integral J_2,

$$\begin{gathered} \int_{u_1=0}^{1} \int_{u_2=u_1-1}^{0} \frac{e^{jk\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}}}{4\pi\sqrt{\alpha_1 u_1^2 + 2\alpha_2 u_1 u_2 + \alpha_3 u_2^2}} \cdots du_2 du_1 = \\ \int_{\omega=0}^{1} \int_{\eta=0}^{1} \frac{e^{jk\sqrt{\alpha_1(\eta\omega)^2 + 2\alpha_2\eta\omega^2(\eta-1) + \alpha_3\omega^2(\eta-1)^2}}}{4\pi\sqrt{\alpha_1(\eta\omega)^2 + 2\alpha_2\eta\omega^2(\eta-1) + \alpha_3\omega^2(\eta-1)^2}} \, \omega \cdots d\eta d\omega, \end{gathered} \tag{65}$$

removes the singularity and generates a regular integral. The final term in (55), J_3, has a form similar to that of J_2,

$$J_3 = \int_{u_1=0}^{1} \int_{u_2=u_1}^{1} \cdots \, du_2 du_1 , \tag{66}$$

the Duffy coordinate transformation to use is,

$$\begin{aligned} u_1 &= \eta\omega, \\ u_2 &= \omega. \end{aligned} \tag{67}$$

The three separate Duffy coordinate transformations have projected the relative coordinate integration in the J integrals onto the same domain. This is the first step in combining the three J terms together so that can be joined together in one integral.

Analytic Integration over ξ_1, ξ_2

The inner integration over the $\bar{\xi}$ coordinate in (50), (53) and (54) can be performed exactly since the integrand depending on these coordinates can be expressed as a simple polynomial. Expanding and summing the various terms in the integrand produces the following expression,

$$\frac{1}{4}\left\{h(\bar{\xi},\bar{u})+h(\bar{\xi}+\bar{u},-\bar{u})\right\}=a_{15}\xi_1^2+a_{14}\xi_2^2+a_{13}\xi_1\xi_2+a_{12}\xi_1+a_{11}\xi_2+a_{10}u_1^2+a_9u_2^2+a_8u_1u_2$$
$$+a_7u_1+a_6u_2+a_5\xi_1u_1+a_4\xi_1u_2+a_3\xi_2u_1+a_2\xi_2u_2+a_1. \tag{68}$$

The coefficients of the polynomial are,

$$\begin{array}{lllll} a_{15}=2\beta_1 & a_{14}=2\beta_4 & a_{13}=2\beta_2+2\beta_3 & a_{12}=2\beta_5+2\beta_7 & a_{11}=2\beta_6+2\beta_8 \\ a_{10}=-\beta_1 & a_9=-\beta_4 & a_8=-\beta_2-\beta_3 & a_7=\beta_5 & a_6=\beta_6 \\ a_5=\beta_1 & a_4=\beta_3 & a_3=\beta_2 & a_2=\beta_4 & a_1=\beta_9-4/k^2 \end{array} \tag{69}$$

where the β's are computed from (42). The analytic integration in the three J integrals require the following six basic forms,

$$K^0(a,b,c,d)=\int_{\xi_1=a}^{b}\int_{\xi_2=c}^{\xi_1+d}d\xi_2d\xi_1=\frac{b^2-a^2}{2}+\frac{b-a}{1}(d-c) \tag{70}$$

$$K^1(a,b,c,d)=\int_{\xi_1=a}^{b}\int_{\xi_2=c}^{\xi_1+d}\xi_1d\xi_2d\xi_1=\frac{b^3-a^3}{3}+\frac{b^2-a^2}{2}(d-c) \tag{71}$$

$$K^2(a,b,c,d)=\int_{\xi_1=a}^{b}\int_{\xi_2=c}^{\xi_1+d}\xi_1^2d\xi_2d\xi_1=\frac{b^4-a^4}{4}+\frac{b^3-a^3}{3}(d-c) \tag{72}$$

$$K^3(a,b,c,d)=\int_{\xi_1=a}^{b}\int_{\xi_2=c}^{\xi_1+d}\xi_2d\xi_2d\xi_1=\frac{(b+d)^3-(a+d)^3}{6}+\frac{c^2}{2}(b-a) \tag{73}$$

$$K^4(a,b,c,d)=\int_{\xi_1=a}^{b}\int_{\xi_2=c}^{\xi_1+d}\xi_2^2d\xi_2d\xi_1=\frac{(b+d)^4-(a+d)^4}{12}+\frac{c^3}{3}(b-a) \tag{74}$$

$$K^5(a,b,c,d)=\int_{\xi_1=a}^{b}\int_{\xi_2=c}^{\xi_1+d}\xi_2\xi_1d\xi_2d\xi_1=\frac{b^4-a^4}{12}+dK^2(a,b,c,d)+\frac{d^2-c^2}{2}K^1(a,b,c,d) \tag{75}$$

The inner integration in the J_1 integral is,

$$\int_{\xi_1=0}^{1-u_1}\int_{\xi_2=0}^{\xi_1}\left\{h(\bar{\xi},\bar{u})+h(\bar{\xi}+\bar{u},-\bar{u})\right\}d\xi_2d\xi_1 \tag{76}$$

let this be denoted as $L_1(u_1,u_2)$ and because it can be integrated analytically, be re-written as,

$$
\begin{gathered}
a_{15}K^2(0,1-u_1,0,0)+a_{14}K^4(0,1-u_1,0,0)+a_{13}K^5(0,1-u_1,0,0)+ \\
(a_{12}+a_5u_1+a_4u_2)K^1(0,1-u_1,0,0)+(a_{11}+a_3u_1+a_2u_2)K^3(0,1-u_1,0,0)+ \\
(a_{10}u_1^2+a_9u_2^2+a_8u_1u_2+a_7u_1+a_6u_2+a_1)K^0(0,1-u_1,0,0).
\end{gathered} \tag{77}
$$

Likewise, denote the J_2 inner integration as $L_2(u_1,u_2)$ which is equal to,

$$
\begin{gathered}
a_{15}K^2(-u_2,1-u_1,-u_2,0)+a_{14}K^4(-u_2,1-u_1,-u_2,0)+a_{13}K^5(-u_2,1-u_1,-u_2,0)+ \\
(a_{12}+a_5u_1+a_4u_2)K^1(-u_2,1-u_1,-u_2,0)+(a_{11}+a_3u_1+a_2u_2)K^3(-u_2,1-u_1,-u_2,0)+ \\
(a_{10}u_1^2+a_9u_2^2+a_8u_1u_2+a_7u_1+a_6u_2+a_1)K^0(-u_2,1-u_1,-u_2,0),
\end{gathered} \tag{78}
$$

and the J_3 inner integration as $L_3(u_1,u_2)$ which is equal to,

$$
\begin{gathered}
a_{15}K^2(u_2-u_1,1-u_1,0,u_2-u_1)+a_{14}K^4(u_2-u_1,1-u_1,0,u_2-u_1)+a_{13}K^5(u_2-u_1,1-u_1,0,u_2-u_1) \\
+(a_{12}+a_5u_1+a_4u_2)K^1(u_2-u_1,1-u_1,0,u_2-u_1)+(a_{11}+a_3u_1+a_2u_2)K^3(u_2-u_1,1-u_1,0,u_2-u_1) \\
+(a_{10}u_1^2+a_9u_2^2+a_8u_1u_2+a_7u_1+a_6u_2+a_1)K^0(u_2-u_1,1-u_1,0,u_2-u_1).
\end{gathered} \tag{79}
$$

This allows the original singular integral to written in compact form as a regular integral of the following form,

$$
\begin{gathered}
\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\eta_1 d\xi_2 d\xi_1 = \\
\frac{1}{4}\int_{\omega=0}^{1}\int_{\eta=0}^{1} \left\{ L_1\times g(\omega,\omega\eta)+L_2\times g(\omega\eta,\omega(\eta-1))+L_3\times g(\omega\eta,\omega) \right\}\omega\, d\eta\, d\omega.
\end{gathered} \tag{80}
$$

In (80) the appropriate Duffy transformations are used in each part of the integrand to map the relative coordinate $\bar{u}=(u_1,u_2)$ to (ω,η) space.

Analytic Integration over ω

The integrand in (80) contains terms of the form, $\omega^n e^{-jk\omega R(\bar{z})}$, and can be integrated analytically with respect to ω. The algebraic complexity of the analytical integration is such that it is prudent to utilize a modern computer algebra system (CAS) to perform this tedious work. In addition, these CAS tools can generate Fortran or C code directly and aid immensely in implementing these results into existing or new computer codes.

The remaining integration over η, must be performed numerically. However, the analysis presented in this section has transformed the original 4-dimensional singular integral into a one-dimensional non-singular numerical integration.

Common Edge

The instance where the two triangular facets share a common edge is formulated so the two triangles share the vertices, $\vec{\mathrm{V}}_1$ and $\vec{\mathrm{V}}_2$. The relative distance vector,

$$\begin{aligned}\vec{r}-\vec{r}' &= (\eta_1-\xi_1)\vec{\mathrm{V}}_1+(\eta_2-\xi_2-\eta_1+\xi_1)\vec{\mathrm{V}}_2+\xi_1\vec{\mathrm{V}}_3-\eta_1\vec{\mathrm{V}}_3' \\ &= u_1(\vec{\mathrm{V}}_1-\vec{\mathrm{V}}_2-\vec{\mathrm{V}}_3')+u_2(\vec{\mathrm{V}}_2)+\xi_1(\vec{\mathrm{V}}_3-\vec{\mathrm{V}}_3')\end{aligned} \tag{81}$$

is a function of the three coordinates, u_1, u_2 and ξ_1. The scalar distance $R=|\vec{r}-\vec{r}'|$, is zero only when $u_1=u_2=\xi_1=0$. In order to remove the singularity that arises in this case, a 3 dimensional Duffy transform is used in the coordinate system defined by u_1, u_2 and ξ_1.

The original 4-dimensional integral (9) in simplex coordinates was decomposed into a sum of 6 separate integrals,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1}\frac{f(\overline{\xi},\overline{\eta})}{R(\overline{\xi},\overline{\eta})}d\eta_2 d\eta_1 d\xi_2 d\xi_1=\sum_{i=1}^{6}\mathrm{E}_i\,. \tag{82}$$

In each of the six integrals a 3-dimensional Duffy transform will be applied and the six separate integrals will be summed together to complete the transformation of the original singular integral into a non singular integral of lower order.

Starting with the integral,

$$\mathrm{E}_1=\int_{u_1=-1}^{0}\int_{u_2=u_1}^{0}\int_{\xi_1=-u_1}^{1}\int_{\xi_2=-u_2}^{\xi_1-(u_2-u_1)}\frac{1}{4}h(\overline{\xi},\overline{u})g(\overline{u},\xi_1)d\xi_2 d\xi_1 du_2 du_1\,, \tag{83}$$

let $\vec{z}=-\vec{u}$, and noting that $g(\overline{u},\xi_1)$ does not depend on ξ_2, express E_1 as,

$$\mathrm{E}_1=\int_{z_1=0}^{1}\int_{z_2=0}^{z_1}\int_{\xi_1=z_1}^{1}g(-\overline{z},\xi_1)\int_{\xi_2=z_2}^{\xi_1-(z_1-z_2)}\frac{1}{4}h(\overline{\xi},-\overline{z})d\xi_2 d\xi_1 dz_2 dz_1\,. \tag{84}$$

The inner integral, over ξ_2, can be performed analytically since h is a simple polynomial in ξ_2. The domain of integration of the remaining 3 dimensional integral is shown in Figure 4, it is a tetrahedral with one point at the origin. Equation (84) is not in a form for a direct Duffy transformation, and must be altered. The order of the integration can be changed in a cyclic manner, with the aid of Figure 4, so that E_1 can be re-written as,

$$\mathrm{E}_1 = \int_{\xi_1=0}^{1} \int_{z_1=0}^{\xi_1} \int_{z_2=0}^{z_1} g(-\bar{z},\xi_1) \int_{\xi_2=z_2}^{\xi_1-(z_1-z_2)} \frac{1}{4} h(\bar{\xi},-\bar{z}) d\xi_2 dz_2 dz_1 d\xi_1 . \tag{85}$$

Equation (85) is now in a form for a direct Duffy transform. The required coordinate transform can be written directly,

$$\begin{pmatrix} \xi_1 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega\, x_1 \\ \omega\, x_1\, x_2 \end{pmatrix}. \tag{86}$$

The Jacobian of the transformation is $\omega^2 x_1$. Equation (85) now can be written in standard form where the coordinates of the g and h functions are expressed in terms of the original u and ξ coordinates,

$$\int_{\omega=0}^{1} \int_{x_1=0}^{1} \int_{x_2=0}^{1} g(-\omega x_1,-\omega x_1 x_2,\omega) \int_{\xi_2=\omega x_1 x_2}^{\omega(1-x_1+x_1x_2)} \frac{1}{4} h(-\omega x_1,-\omega x_1 x_2,\omega,\xi_2) d\xi_2\, \omega^2 x_1 dx_2 dx_1 d\omega . \tag{87}$$

The integral E_2 is treated in a similar fashion,

$$\mathrm{E}_2 = \int_{u_1=0}^{1} \int_{u_2=0}^{u_1} \int_{\xi_1=0}^{1-u_1} \int_{\xi_2=0}^{\xi_1} \frac{1}{4} h(\bar{\xi},\bar{u}) g(\bar{u},\xi_1) d\xi_2 d\xi_1 du_2 du_1 , \tag{88}$$

where the substitution, $r = \xi_1 + u_1$, produces,

$$\mathrm{E}_2 = \int_{u_1=0}^{1} \int_{u_2=0}^{u_1} \int_{r=u_1}^{1} g(\bar{u},r) \int_{\xi_2=0}^{r-u_1} \frac{1}{4} h(r,\xi_2,\bar{u}) d\xi_2 dr du_2 du_1 . \tag{89}$$

The order of integration is altered similar to (85),

$$\mathrm{E}_2 = \int_{r=0}^{1} \int_{u_1=0}^{r} \int_{u_2=0}^{u_1} g(\bar{u},r) \int_{\xi_2=0}^{r-u_1} \frac{1}{4} h(r,\xi_2,\bar{u}) d\xi_2 du_2 du_1 dr . \tag{90}$$

The 3-dimensional Duffy coordinate transformation for this integral is,

$$\begin{pmatrix} r \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega\, x_1 \\ \omega\, x_1\, x_2 \end{pmatrix}. \tag{91}$$

Using (91) the final form of E_2 is,

$$\int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(\omega x_1,\omega x_1 x_2,\omega(1-x_1))\int_{\xi_2=0}^{\omega(1-x_1)}\frac{1}{4}h(\omega x_1,\omega x_1 x_2,\omega(1-x_1),\xi_2)d\xi_2\omega^2 x_1 dx_2 dx_1 d\omega\,. \tag{92}$$

The third integral,

$$E_3=\int_{u_1=-1}^{0}\int_{u_2=0}^{1+u_1}\int_{\xi_1=u_2-u_1}^{1}\int_{\xi_2=0}^{\xi_1-(u_2-u_1)}\frac{1}{4}h(\bar{\xi},\bar{u})g(\bar{u},\xi_1)d\xi_2 d\xi_1 du_2 du_1\,, \tag{93}$$

is transformed by substituting, $z_1=-u_1, z_2=u_2$,

$$E_3=\int_{z_1=0}^{1}\int_{z_2=0}^{1-z_1}\int_{\xi_1=z_2+z_1}^{1} g(\bar{z},\xi_1)\int_{\xi_2=0}^{\xi_1-(z_2+z_1)}\frac{1}{4}h(\bar{\xi},\bar{z})d\xi_2 d\xi_1 dz_2 dz_1\,, \tag{94}$$

and $r=z_1+z_2$,

$$E_3=\int_{z_1=0}^{1}\int_{r=z_1}^{1}\int_{\xi_1=r}^{1} g(z_1,r,\xi_1)\int_{\xi_2=0}^{\xi_1-(z_2+z_1)}\frac{1}{4}h(z_1,r,\xi_1,\xi_2)d\xi_2 d\xi_1 dr dz_1\,. \tag{95}$$

The integration domain for (95) is shown in Figure 5, and it is slightly different than the domain of E_1. The domain is a tetrahedral with one corner at the origin but it is oriented along the r axis. The cyclic reordering of the integration over this domain lets E_3 be re-written as,

$$E_3=\int_{\xi_1=0}^{1}\int_{r=0}^{\xi_1}\int_{z_1=0}^{r} g(z_1,r,\xi_1)\int_{\xi_2=0}^{\xi_1-(z_2+z_1)}\frac{1}{4}h(z_1,r,\xi_1,\xi_2)d\xi_2 dz_1 dr d\xi_1\,. \tag{96}$$

The 3-dimensional Duffy coordinate transformation for this integral is,

$$\begin{pmatrix}\xi_1\\ r\\ z_1\end{pmatrix}=\begin{pmatrix}\omega\\ \omega x_1\\ \omega x_1 x_2\end{pmatrix}. \tag{97}$$

E_3 in final form, written with g and h as functions of the original simplex and relative coordinates, becomes,

$$\int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(-\omega x_1 x_2,\omega(1-x_1),\omega)\int_{\xi_2=0}^{\omega(1-x_1)}\frac{1}{4}h(-\omega x_1 x_2,\omega(1-x_1),\omega,\xi_2)d\xi_2\omega^2 x_1 dx_2 dx_1 d\omega\,. \tag{98}$$

The remaining integrals are treated in a similar fashion and the results are summarized here.

$$E^1 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(-\omega x_1, -\omega x_1 x_2, \omega) h^1(\omega, x_1, x_2)\, \omega^2 x_1 dx_2 dx_1 d\omega\,, \tag{99}$$

$$E^2 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(\omega x_1, \omega x_1 x_2, \omega(1-x_1)) h^2(\omega, x_1, x_2)\, \omega^2 x_1 dx_2 dx_1 d\omega\,, \tag{100}$$

$$E^3 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(-\omega x_1 x_2, \omega(1-x_1), \omega) h^3(\omega, x_1, x_2)\, \omega^2 x_1 dx_2 dx_1 d\omega\,, \tag{101}$$

$$E^4 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(\omega x_1 x_2, \omega x_1(1-x_2), \omega(1-x_1 x_2)) h^4(\omega, x_1, x_2)\, \omega^2 x_1 dx_2 dx_1 d\omega\,, \tag{102}$$

$$E^5 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(-\omega x_1 x_2, -\omega x_1, \omega) h^5(\omega, x_1, x_2)\, \omega^2 x_1 dx_2 dx_1 d\omega\,, \tag{103}$$

$$E^6 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} g(\omega x_1, \omega x_1 x_2, \omega(1-x_1)) h^6(\omega, x_1, x_2)\, \omega^2 x_1 dx_2 dx_1 d\omega\,. \tag{104}$$

The common edge Galerkin inner product 4-dimensional integral (82) can now be expressed as a single integral with a regular integrand,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\eta_1 d\xi_2 d\xi_1 = \int_{\omega=0}^{1}\int_{x_1=0}^{1}\int_{x_2=0}^{1} \sum_{i=1}^{6} g_i h^i\, \omega^2 x_1\, dx_2 dx_1 d\omega\,. \tag{105}$$

Analytic Integration over ξ_2

The integrand of (105) is the product of the Greens function, the Jacobian of the 3-dimensional Duffy transform and a series of polynomials. The exact form of the polynomials, h^i, are derived from the following analytic integrals.

$$h^1(\omega, x_1, x_2) = \int_{\xi_2=\omega x_1 x_2}^{\omega(1-x_1+x_1 x_2)} \frac{1}{4} h(-\omega x_1, -\omega x_1 x_2, \omega, \xi_2) d\xi_2 \tag{106}$$

$$h^2(\omega, x_1, x_2) = \int_{\xi_2=0}^{\omega(1-x_1)} \frac{1}{4} h(\omega x_1, \omega x_1 x_2, \omega(1-x_1), \xi_2) d\xi_2 \tag{107}$$

$$h^3(\omega, x_1, x_2) = \int_{\xi_2=0}^{\omega(1-x_1)} \frac{1}{4} h(-\omega x_1 x_2, \omega(1-x_1), \omega, \xi_2) d\xi_2 \tag{108}$$

$$h^4(\omega,x_1,x_2)=\int_{\xi_2=\omega x_1(1-x_2)}^{\omega(1-x_1x_2)}\frac{1}{4}h(\omega x_1x_2,\omega x_1(1-x_2),\omega(1-x_1x_2),\xi_2)d\xi_2 \tag{109}$$

$$h^5(\omega,x_1,x_2)=\int_{\xi_2=\omega x_1}^{\omega}\frac{1}{4}h(-\omega x_1x_2,-\omega x_1,\omega,\xi_2)d\xi_2 \tag{110}$$

$$h^6(\omega,x_1,x_2)=\int_{\xi_2=0}^{\omega(1-x_1x_2)}\frac{1}{4}h(\omega x_1,\omega x_1x_2,\omega(1-x_1),\xi_2)d\xi_2 \tag{111}$$

The integrand of the h^i expressions is given by,

$$\begin{aligned}h(u_1,u_2,\xi_1,\xi_2)&=\vec{\rho}_j\cdot\vec{\rho}_k-4/k^2\\&=\beta_1\xi_1(u_1+\xi_1)+\beta_2\xi_1(u_2+\xi_2)+\beta_3\xi_2(u_1+\xi_1)+\beta_4(u_2+\xi_2)\xi_2\\&\quad+\beta_5\xi_1+\beta_6\xi_2+\beta_7(u_1+\xi_1)+\beta_8(u_2+\xi_2)+\beta_9-4/k^2\end{aligned} \tag{112}$$

In general, the h^i can be written in this form,

$$\begin{aligned}h^i(\omega,x_1,x_2)&=\int_a^b\frac{1}{4}h(u_1,u_2,\xi_1,\xi_2)d\xi_2\\&=\int_a^b\alpha_1\xi_2^2+\alpha_2\xi_2+\alpha_3d\xi_2\\&=(b-a)\left\{\alpha_1\frac{b^2+a^2}{3}+\alpha_2\frac{b+a}{2}+\alpha_3\right\}\end{aligned} \tag{113}$$

where,

$$\begin{aligned}\alpha_1&=\beta_4\\\alpha_2&=\beta_3u_1+\beta_4u_2+(\beta_2+\beta_3)\xi_1+\beta_6+\beta_8\\\alpha_3&=(\beta_1\xi_1+\beta_7)u_1+(\beta_2\xi_1+\beta_8)u_2+(\beta_5+\beta_7)\xi_1+\beta_1\xi_1^2+\beta_9-4/k^2\end{aligned} \tag{114}$$

the β_i are given in (42). In (105) the integration over ω can be performed analytically because the integrand contains terms of the form, $\omega^n e^{-jk\omega R(\Xi)}$. The final result is that 2 of the integrations in the original 4 dimensional integral can be performed analytically.

Common Vertex

The case where the two triangles share a common vertex is a less complex problem. If the common vertex is chosen to be V_1 then this point is mapped to the origin in the simplex coordinate space. The singularity arising from the Greens function spatial distance term,

$$R = |\vec{r} - \vec{r}'|,$$

expressed in the simplex coordinates becomes,

$$\vec{r} - \vec{r}' = (\eta_1 - \xi_1)\vec{\mathrm{V}}_1 + (\xi_1 - \xi_2)\vec{\mathrm{V}}_2 + \xi_1\vec{\mathrm{V}}_3 - (\eta_1 - \eta_2)\vec{\mathrm{V}}_2' - \eta_1\vec{\mathrm{V}}_3'. \tag{115}$$

Equation (115) is zero only for,

$$\eta_1 = \eta_2 = \xi_1 = \xi_2 = 0,$$

which is the origin of the 4 dimensional simplex coordinate space. The integral,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\eta_1 d\xi_2 d\xi_1, \tag{116}$$

with the isolated singularity at the origin can be regularized directly by using a 4 dimensional generalized Duffy transformation to eliminate this singularity. In order to accomplish this, the order of the integration in (116) is first changed slightly,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\eta_1 d\xi_2 d\xi_1 = \int_{\xi_1=0}^{1}\int_{\eta_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\eta_1 d\xi_1, \tag{117}$$

and the domain split into two parts,

$$\begin{aligned} &\int_{\xi_1=0}^{1}\int_{\eta_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\eta_1 d\xi_1 = \\ &\int_{\xi_1=0}^{1}\int_{\eta_1=0}^{\xi_1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\eta_1 d\xi_1 + \int_{\xi_1=0}^{1}\int_{\eta_1=\xi_1}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\eta_1 d\xi_1. \end{aligned} \tag{118}$$

In the second integral,

$$\int_{\xi_1=0}^{1}\int_{\eta_1=\xi_1}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\eta_1 d\xi_1, \tag{119}$$

reverse the order of the η_1, ξ_1, integration,

$$\int_{\xi_1=0}^{1}\int_{\eta_1=\xi_1}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\eta_1 d\xi_1 = \int_{\eta_1=0}^{1}\int_{\xi_1=0}^{\eta_1}\int_{\xi_2=0}^{\eta_1}\int_{\eta_2=0}^{\xi_1} \frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})} d\eta_2 d\xi_2 d\xi_1 d\eta_1, \tag{120}$$

which is identical to the first integral under a coordinate interchange, $(\bar{\xi},\bar{\eta}) \Rightarrow (\bar{\eta},\bar{\xi})$. The common vertex integral can now be written as,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1}\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}d\eta_2 d\eta_1 d\xi_2 d\xi_1 = \int_{\xi_1=0}^{1}\int_{\eta_1=0}^{\xi_1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_2=0}^{\eta_1}\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}+\frac{f(\bar{\eta},\bar{\xi})}{R(\bar{\eta},\bar{\xi})}d\eta_2 d\xi_2 d\eta_1 d\xi_1 . \tag{121}$$

The Duffy transformation needed to eliminate the singularity at the origin in this integral has the form,

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}\omega\\ \omega z_1\end{pmatrix}, \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\omega z_2\\ \omega z_2 z_3\end{pmatrix}, \tag{122}$$

This produces the following result,

$$\int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\int_{\eta_1=0}^{1}\int_{\eta_2=0}^{\eta_1}\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}d\eta_2 d\eta_1 d\xi_2 d\xi_1 = \int_{\omega=0}^{1}\int_{z_1=0}^{1}\int_{z_2=0}^{1}\int_{z_3=0}^{1}\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}+\frac{f(\bar{\eta},\bar{\xi})}{R(\bar{\eta},\bar{\xi})}\omega^3 z_2\, dz_3 dz_2 dz_1 d\omega. \tag{123}$$

The integrand has been left written in terms of the simplex coordinates because of the coordinate interchange. To simplify the integrand is expressed as,

$$\frac{f(\bar{\xi},\bar{\eta})}{R(\bar{\xi},\bar{\eta})}=\left(\frac{\vec{\rho}_j^{\,p}(\bar{\xi})\cdot\vec{\rho}_k^{\,q}(\bar{\eta})}{4}-\frac{1}{k^2}\right)g(\bar{\xi},\bar{\eta})=\frac{1}{4}h(\bar{\xi},\bar{\eta})g(\bar{\xi},\bar{\eta}). \tag{124}$$

The first term, $h(\bar{\xi},\bar{\eta})$, can be expressed as a polynomial,

$$h(\bar{\xi},\bar{\eta})=a_1\xi_1\eta_1+a_2\xi_1\eta_2+a_3\xi_2\eta_1+a_4\eta_2\xi_2+a_5\xi_1+a_6\xi_2+a_7\eta_1+a_8\eta_2+a_9 \tag{125}$$

The Greens function depends only on the distance,

$$\begin{aligned}\vec{r}-\vec{r}\,' &= (\eta_1-\xi_1)\vec{\mathrm{V}}_1+(\xi_1-\xi_2)\vec{\mathrm{V}}_2+\xi_1\vec{\mathrm{V}}_3-(\eta_1-\eta_2)\vec{\mathrm{V}}_2'-\eta_1\vec{\mathrm{V}}_3',\\ &= \vec{a}\xi_1+\vec{b}\,\xi_2+\vec{c}\,\eta_1+\vec{d}\eta_2,\end{aligned} \tag{126}$$

where,

$$\vec{a}=\vec{\mathrm{V}}_2-\vec{\mathrm{V}}_1+\vec{\mathrm{V}}_3,\ \vec{b}=-\vec{\mathrm{V}}_2,\ \vec{c}=\vec{\mathrm{V}}_1-\vec{\mathrm{V}}_2'-\vec{\mathrm{V}}_3',\ \vec{d}=\vec{\mathrm{V}}_2'. \tag{127}$$

The scalar distance function is expanded in a polynomial,

$$R(\bar{\xi},\bar{\eta})=\sqrt{\left(\vec{a}\xi_1+\vec{b}\,\xi_2+\vec{c}\,\eta_1+\vec{d}\eta_2\right)\cdot\left(\vec{a}\xi_1+\vec{b}\,\xi_2+\vec{c}\,\eta_1+\vec{d}\eta_2\right)}, \tag{128}$$

in compact form,

$$R(\bar{\xi},\bar{\eta})=\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_{i,j}\xi_i\xi_j+\beta_{i,j}\xi_i\eta_j+\chi_{i,j}\xi_j\eta_i+\delta_{i,j}\eta_i\eta_j\right)^{1/2}. \tag{129}$$

The form with the coordinate exchange is,

$$R(\bar{\eta},\bar{\xi})=\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_{i,j}\eta_i\eta_j+\beta_{i,j}\xi_j\eta_i+\chi_{i,j}\xi_i\eta_j+\delta_{i,j}\xi_i\xi_j\right)^{1/2}. \tag{130}$$

The Duffy coordinate transform introduces a ω into each of the simplex coordinates. The distance function polynomial in the Greens function allows a single ω to be brought out from under the square root. The Jacobian of the transform cancels this term from the denominator of the Greens function and eliminates the singularity.

The original integral over the pair of triangular facets with a common vertex is now written as,

$$\frac{1}{4}\int_{\omega=0}^{1}\int_{z_1=0}^{1}\int_{z_2=0}^{1}\int_{z_3=0}^{1}\left(h(\bar{\xi},\bar{\eta})g(\bar{\xi},\bar{\eta})+h(\bar{\eta},\bar{\xi})g(\bar{\eta},\bar{\xi})\right)\omega^3 z_2\,dz_3dz_2dz_1d\omega. \tag{131}$$

The integration over ω can be performed analytically because the integrand contains terms of the form, $\omega^n e^{-jk\omega R(\bar{z})}$. The remaining integrations must be performed numerically.

Summary and Conclusion

The Galerkin solution to the electric field integral equation using RWG basis functions has and will continue to be popular for simulating electromagnetic phenomena. Fundamental to achieving an accurate result is the numerical accuracy of the impedance matrix elements that are used in computing the solution. The material presented here shows how to improve the computation of matrix elements that involve singular surface integrals by removing the singularity in the integrand through a series of coordinate transforms. The formulation provided an additional benefit by reducing the dimension of the 4-dimensional integrations that result from the Galerkin approach by 1, 2 or 3 depending on the degree of overlap between the two triangular patches.

This analysis can be extended to curved geometries, quadrilateral patch geometry and other integral equation operators encountered in electromagnetics. The extension of these methods to hypersingular integral equation operators will provide a complete foundation for highly accurate method of moments numerical solutions.

The material presented here is a modernization of the classic RWG method. The next step in this modernization process is to utilize computer algebra software, e.g. Maple or Mathematica, to complete the tedious algebraic manipulations and analytic integrations. The ultimate goal is to generate modular Fortran90/95 code that could be used as a foundation for modern method of moments (MoM) electromagnetic analysis tools.

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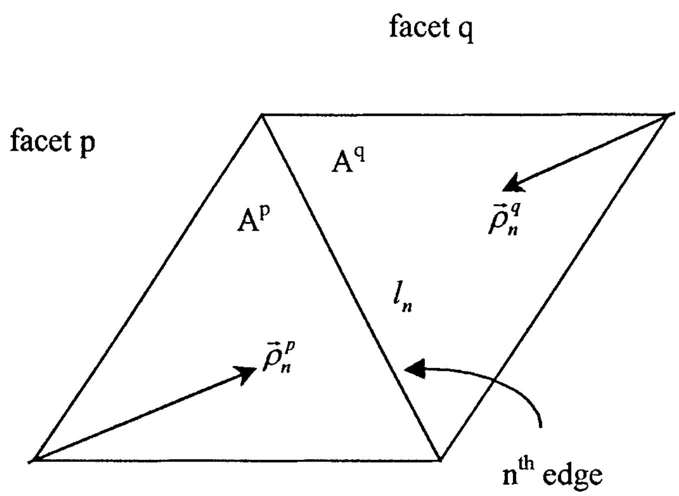

$$\vec{f}(\vec{r}) = \begin{cases} \dfrac{S_p^+ l_n}{2A_p}\,\vec{\rho}_n^{\,p};\ S_p^+ = +1 \\ \dfrac{S_q^- l_n}{2A_q}\,\vec{\rho}_n^{\,q};\ S_q^+ = -1 \end{cases}$$

Figure 1. Rao, Wilton, Glisson (RWG) vector basis function. The basis function consists of a pair of triangular patches, with areas A_p and A_q, which share a common edge with length l_n. The expansion of the unknown surface electric current in terms of the vector function $\vec{f}(\vec{r})$ is associated with the n^{th} edge.

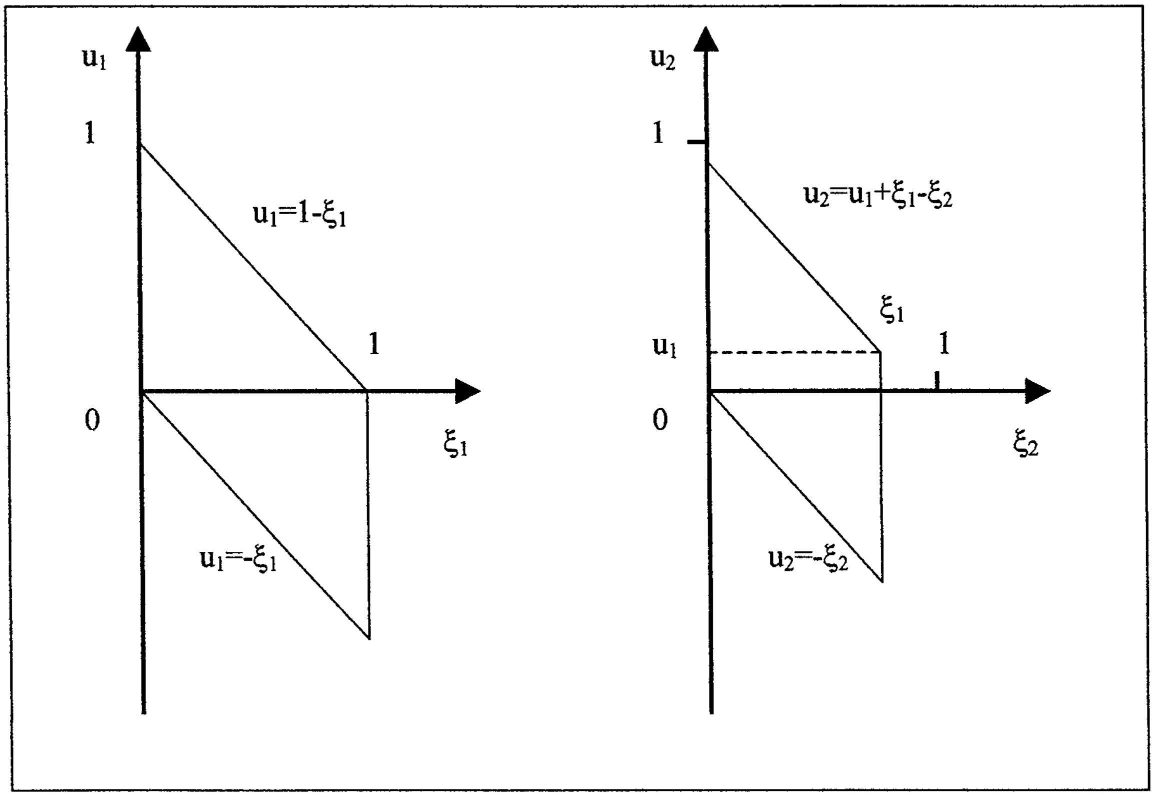

Figure 2. The domain of integration of the 4 dimensional Galerkin inner product integral expressed in terms of relative and simplex coordinates. This is used for interchanging the order of integration in the integrals I^1 through I^6.

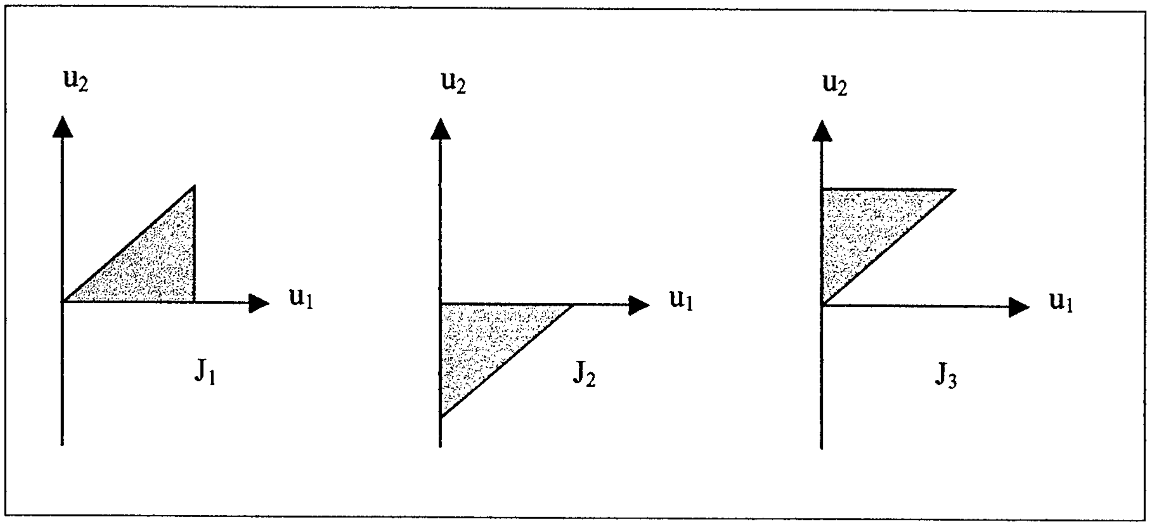

Figure 3. The domain of integration in relative coordinates for the three integrals, J_1, J_2 and J_3, that result from the common facet case.

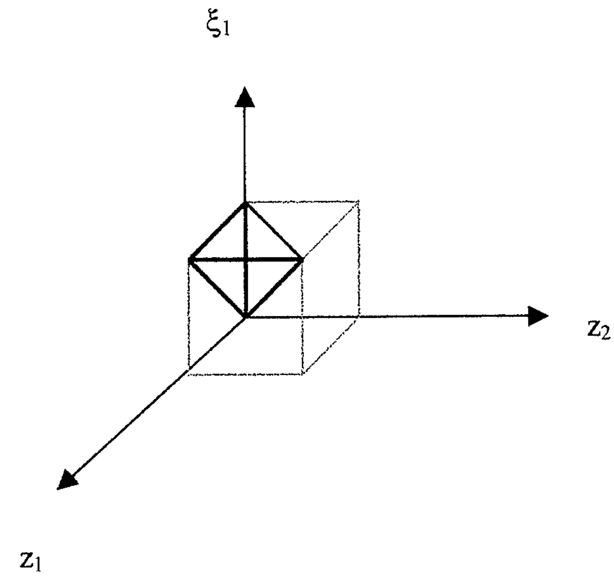

Figure 4. The domain of integration of the integral E_1 for the common edge case. A unit cube is shown for reference, the red lines define a tetrahedral region which border the domain of integration of the E_1 integral.

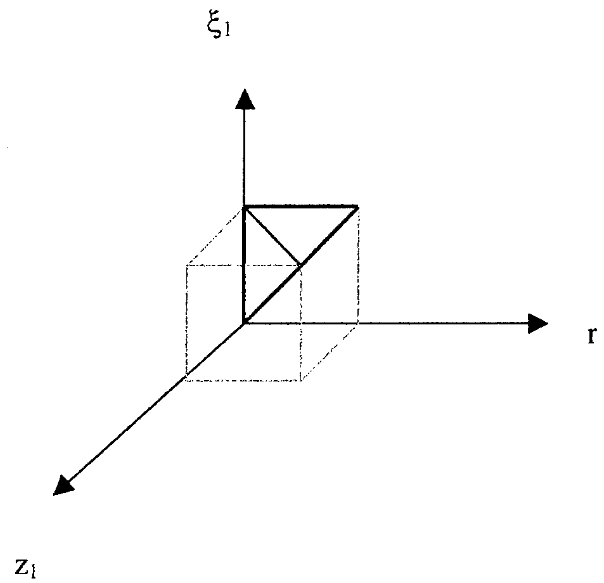

Figure 5. The domain of integration of the integral E_3 for the common edge case. A unit cube is shown for reference, the red lines define a tetrahedral region which border the volume of integration of the E_3 integral.